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Optimization and Differential Equations in FRQ’s
AP Readiness Session 8 - March

Answers to examples posted on my website

Optimization General Problem Steps

- Find the rate of change of whatever is being optimized. Sometimes that rate of change is given as a graph. Sometimes it is given as function. Sometimes you will to differentiate a given function.
- Set the rate of change = 0 to find critical points on a given interval. Sometimes this requires the use of a calculator.
- Substitute all critical points and endpoints into the function that is being optimized to compare results and find the absolute maximum or minimum.

Differential Equation General Problem Steps

Solving a differential equation
- Separate the variable
- Integrate each side
- Include a constant of integration on one side
- Substitute given initial values for both variables
- Solve for the constant of integration
- Substitute the constant of integration value into the equation and solve for the function variable

Other types of subproblems
- Slope fields – substitute given values and sketch slopes
- Tangent lines – substitute given values to find slope
- Second derivative – be careful to differentiate implicitly

Examples

1.
For $0 \le t \le 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.
To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 31$? Show the analysis that leads to your conclusion.

2.
A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ($t = 0$) and 8 P.M. ($t = 8$). The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

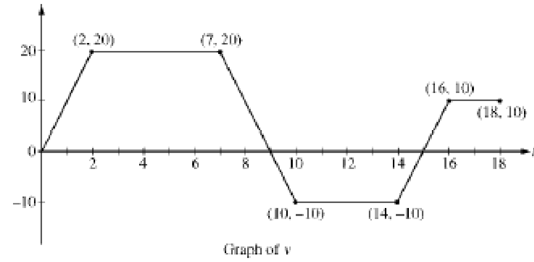
t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, at what time were the entries being processed most quickly?

3.

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at this time?



Graph of v

4.

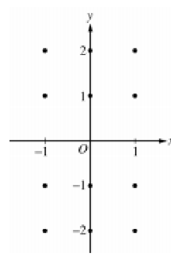
Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

(Note: Use the axes provided in the exam booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.



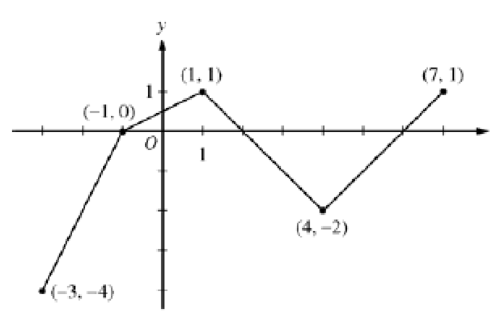
5.

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

Team Practice
1. A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$. Find the time t at which the particle is farthest to the left. Justify your answer.
2. The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$. Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.
3. The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by $$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$ A pumping station adds sand to the beach at a rate modeled by the function S, given by $$S(t) = \frac{15t}{1+3t}.$$ Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand. For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.
4. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$. Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.  Graph of g'

5.

Let f be a function with $f(2) = -8$ such that for all points (x, y) on the graph of f, the slope is given by $\frac{3x^2}{y}$.

(a) Write an equation of the line tangent to the graph of f at the point where $x = 2$ and use it to approximate $f(1.8)$.

(b) Find an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = \frac{3x^2}{y}$ with the initial condition $f(2) = -8$.

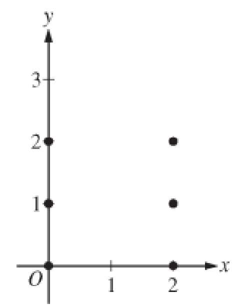
6.

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.



7.

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.